

ON PRYM VARIETIES FOR THE COVERINGS OF SOME SINGULAR PLANE CURVES

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ABSTRACT. Let k be a field of characteristic zero containing a primitive n -th root of unity. Let C_n^0 be a singular plane curve of degree n over k admitting an order n automorphism, n nodes as the singularities, and C_n be its normalization.

In this paper we study the factors of Prym variety $\mathrm{Prym}(\tilde{C}_n/C_n)$ associated to the double cover \tilde{C}_n of C_n exactly ramified at the points obtained by the blow-up of the singularities. We provide explicit models of some algebraic curves related to the construction of $\mathrm{Prym}(\tilde{C}_n/C_n)$ as a Prym variety and determine the interesting simple factors other than elliptic curves or hyperelliptic curves with small genus which come up in J_n so that the endomorphism rings contains the totally real field $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

1. INTRODUCTION

Let k be a field of characteristic zero that contains a primitive n -th root of unity ζ for a positive integer n . Let C_n^0 be a plane curve of degree n in \mathbb{P}_k^2 with an automorphism of order n over k . Assume that C_n^0 admits n nodes $R_0 := \{P_1, \dots, P_n\}$ defined over k as the singularities. Then the geometric genus of C_n^0 is

$$g(C_n^0) = \frac{(n-1)(n-2)}{2} - n = \frac{n^2 - 5n + 2}{2}.$$

Let C_n be the normalization of C_n^0 obtained by blowing up along R_0 . If we write the blow-up for $\pi_0 : C_n \rightarrow C_n^0$ and put $R = \pi_0^{-1}(R_0)$, then R consists of $2n$ points.

We consider the double cover $\pi : \tilde{C}_n \rightarrow C_n$ ramified exactly along R . Then by genus formula we see that

$$g(\tilde{C}_n) = 1 + (2g(C_n) - 2) + \frac{1}{2}|R| = n^2 - 4n + 1$$

since $g(C_n) = g(C_n^0)$ where $g(X)$ stands for the geometric genus of a projective irreducible curve X . Let $\iota : \tilde{C}_n \rightarrow \tilde{C}_n$ be the involution associated to the covering π . Then the endomorphism $1 - \iota$ on $\mathrm{Jac}(\tilde{C}_n)$ annihilates $\mathrm{Jac}(C_n)$. Let $J_n = \mathrm{Prym}(\tilde{C}_n/C_n)$ be the Prym variety associated to the covering π which is defined by

$$J_n := (1 - \iota)\mathrm{Jac}(\tilde{C}_n).$$

which is of dimension $g(\tilde{C}_n) - g(C_n) = \frac{n(n-3)}{2}$. It is well-known that J_n is a principal polarized abelian variety (cf. [Mum]).

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In this paper we focus on the decomposition of J_n over a suitable (finite) extension L of k for some families of plane singular curves with the same properties as C_n^0 . The motivations would be related to Section 2 of [GY] and the splitting of Jacobians (cf. [BSS, Sh, BSh]). The points why we study such a curve C_n^0 are as follows:

- (1) A smooth plane curve of degree d tends to have high genus in its degree and therefore the decomposition of the Jacobian might be hard to analyze. However we can reduce the genus if we consider curves of the same degree d with singularities;
- (2) one can easily find (possibly singular) plane curves which have many automorphisms;
- (3) a smooth low genus curve might not have many automorphisms, but a singular curve with the same geometric genus could have it.

In this vein we explain our main result. Let C_n be a normalization of the singular plane curve in \mathbb{P}_k^2 defined by

$$f_n(x, y, z) := x^n + z^n + (n-2)y^n - nxyz y^{n-2} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i \{ (xz)^i y^{n-2i} - ixzy^{n-2} + (i-1)y^n \} = 0$$

whose singularities are exactly given by n nodes $P_i = (\zeta^i : 1 : \zeta^{-i})$, $0 \leq i \leq n-1$ for generic parameter a_i . Let \tilde{C}_n be the double cover of C_n defined by $\varpi^2 = y^2 - xz$ and $f_n(x, y, z) = 0$ in \mathbb{P}_k^3 . The curve C_n admits the automorphisms

$$\alpha : (x : y : z) \mapsto (\zeta x : y : \zeta^{-1} z), \quad \beta : (x : y : z) \mapsto (z : y : x)$$

and they are naturally extended to \tilde{C}_n which commute with the double cover $\tilde{C}_n \rightarrow C_n$. To abbreviate the notation we write

$$J_{n,\tau} = \text{Prym}(\tilde{C}_n/C_n, \tau) := \text{Prym}((\tilde{C}_n/\langle \tau \rangle)/(C_n/\langle \tau \rangle))$$

for $\tau \in \langle \alpha, \beta \rangle \subset \text{Aut}_k(C_n)$. Then we have a structure theorem in the decomposition of $J_n := \text{Prym}(\tilde{C}_n/C_n)$:

Theorem 1.1. (*Theorem 4.6*) *Let $n = 2^k m \geq 4$ for $k \geq 0$ and odd $m \geq 1$. Assume that C_n is smooth outside n -nodes. Then*

- (1) *If n is odd (hence $k = 0$), then*

$$J_n \stackrel{k}{\sim} J_{n,\alpha} \times J_{n,\beta}^2$$

so that $\text{End}_k(J_{n,\beta}) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

- (2) *If $n \geq 6$ is even, then*

$$J_n \stackrel{k}{\sim} J_{n,\alpha} \times J_{n,\beta} \times J_{n,\beta\alpha^{\frac{n}{m}}}$$

where the latter two Prym varieties are an abelian variety over k whose endomorphism ring contains $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ respectively.

- (3) *If $n = 4$, then*

$$J_4 \stackrel{k}{\sim} J_{4,\beta} \times J_{4,\beta\alpha^2}$$

where the both factors in the right hand side are elliptic curves over k .

This paper is organized as follows. In section 2 we introduce our family of singular plane curves and discuss their automorphism groups. Then, in Section 3 we compute an explicit basis of the curves \tilde{C}_n and \tilde{C}_n . In section 4 we give a proof of Theorem 1.1 according to a main idea invented in [GY]. In section 5 we perform explicit computations for $4 \leq n \leq 8$ and give an explicit decomposition of the corresponding Prym varieties. We also try to realize some of Prym factors as a Jacobian of a curve as long as the computation carries out without any complexity.

Throughout this paper k is a field of characteristic zero containing all n -th roots of unity with infinite cardinality and we denote by D_n the dihedral group of order $2n$, and \tilde{C} the normalization for a projective algebraic curve C .

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2. FAMILIES OF SINGULAR CURVES WITH AUTOMORPHISMS

Let $n \geq 4$. Let $\zeta := \zeta_n$ be a primitive n -th root of unity and k a characteristic zero field such that $k(\zeta_n) = k$. In this paper we consider the following singular plane curve C_n^0 in \mathbb{P}^2 defined over k and given by the equation $f_n(x, y, z) = 0$, where

$$(1) \quad f_n(x, y, z) := x^n + z^n + (n-2)y^n - nxyz^{n-2} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i \{ (xz)^i y^{n-2i} - ixzy^{n-2} + (i-1)y^n \}$$

and $\lfloor \frac{n}{2} \rfloor$ stands for the maximal integer less than or equal to $\frac{n}{2}$ and a_i 's are parameters. The curve C_n^0 has n nodes $P_i := (\zeta^i : 1 : \zeta^{-i})$, $i = 0, \dots, n-1$ and admits two automorphisms:

$$(2) \quad \begin{aligned} \alpha : (x : y : z) &\mapsto (\zeta x : y : \zeta^{-1} z) \\ \beta : (x : y : z) &\mapsto (z : y : x), \end{aligned}$$

defined over k since $\zeta \in k$. Throughout this paper we assume that C_n^0 is smooth at any point other than P_i , $i = 0, \dots, n-1$. One can check this condition would not be so restrictive. In fact when $a_i = 0$ for all $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $k = \mathbb{C}$, we see

$$C_n^0 : x^n + z^n + (n-2)y^n - nxyz^{n-2} = 0$$

which is smooth at any point other than the n -nodes. Hence by continuity our family C_n^0 gives rise to a family whose generic member is smooth other than the n -nodes.

Remark 2.1. In the case of $n = 5$ the genus of C_5^0 is one. On the other hand any projective smooth curve of genus one does not admit an automorphism of order $n = 5$. However C_5^0 does because of the singularities.

Next we consider the double covering in \mathbb{P}^3 defined by:

$$(3) \quad \tilde{C}_n^0 : \begin{cases} w^2 = y^2 - xz \\ f_n(x, y, z) = 0. \end{cases}$$

This curve admits three automorphism

$$(4) \quad \begin{aligned} \alpha &: (x : y : z : w) \mapsto (\zeta x : y : \zeta^{-1} z : w) \\ \beta &: (x : y : z : w) \mapsto (z : y : x : w) \\ \gamma &: (x : y : z : w) \mapsto (x : y : z : -w). \end{aligned}$$

Here we use the same notation for α, β because they are naturally coming from (2). Let \tilde{C}_n be the normalization of \tilde{C}_n^0 and the double cover extends naturally to a double cover $\tilde{C}_n \rightarrow C_n$ which inherits the automorphisms (4). Put $J_n = \text{Prym}(\tilde{C}_n/C_n)$ as before. To study a decomposition of J_n and its factor we make use of the quotient curves by automorphisms.

2.1. Automorphisms. The decomposition of Jacobians of such curves will be realized by investigating the automorphism group of each curve. Hence, we would like to study a specific subgroup in $\text{Aut}_k(C_n^0)$ and $\text{Aut}_k(\tilde{C}_n^0)$ respectively.

Lemma 2.2. *Let C_n^0 and \tilde{C}_n^0 be as above. Then $D_{2n} \hookrightarrow \text{Aut}_k(C_n^0)$ and $\mathbb{Z}/2\mathbb{Z} \times D_{2n} \hookrightarrow \text{Aut}_k(\tilde{C}_n^0)$.*

Proof. It is obvious that α and β have orders n and 2 respectively. Let G be the group generated by α, β . Obviously, $G \hookrightarrow \text{Aut}_k(C_n^0)$. Since $\beta\alpha\beta = \alpha^{-1}$, then G is isomorphic to the dihedral group of order $2n$.

To prove the second part of the theorem we let $G := \langle \alpha, \beta, \gamma \rangle$, where α, β, γ are as in (4). Obviously, α has order n and β and γ are involutions. Since α and β generate a dihedral group and γ commutes with α and β , then they generate a group of order $4n$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times D_{2n}$. \square

2.2. Decomposing the Jacobian by group partitions. Let \mathcal{X} be a genus g algebraic curve defined over k with automorphism group $G = \text{Aut}_k(\mathcal{X})$.

The following result is from [KR, Theorem B]:

Proposition 2.3. *Let $H \subset G$ such that $H = H_1 \cup \dots \cup H_t$ where the subgroups $H_i \subset H$ satisfy $H_i \cap H_j = \{1\}$ for all $i \neq j$. Then, we have the isogeny relation*

$$\text{Jac}(\mathcal{X})^{t-1} \times \text{Jac}(\mathcal{X}/H)^{|H|} \stackrel{k}{\sim} \text{Jac}(\mathcal{X}/H_1)^{|H_1|} \times \dots \times \text{Jac}(\mathcal{X}/H_t)^{|H_t|}$$

As an easy consequence we have

Corollary 2.4. *Keep the notation as above. Assume that $H \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and write $H_0 = \{e\}$ for the trivial subgroup and $H_i \simeq \mathbb{Z}/2\mathbb{Z}$, $1 \leq i \leq 3$ for other three subgroups. Then*

$$\text{Jac}(\mathcal{X}) \times \text{Jac}(\mathcal{X}/H)^2 \stackrel{k}{\sim} \text{Jac}(\mathcal{X}/H_1) \times \text{Jac}(\mathcal{X}/H_2) \times \text{Jac}(\mathcal{X}/H_3).$$

3. HOLOMORPHIC DIFFERENTIAL FORMS

In this section we study holomorphic differential forms on C_n and \tilde{C}_n respectively which will be used to decompose the Prym varieties in question. We work on an affine singular model $\hat{f}_n := f_n(x, 1, z) = 0$ for C_n^0 . Put $\hat{f}_{n,x} := \frac{\partial \hat{f}_n}{\partial x}$ and $\hat{f}_{n,z} := \frac{\partial \hat{f}_n}{\partial z}$.

Theorem 3.1. *The followings hold:*

(1) *The elements*

$$\omega_{r,s} := \frac{(xz-1)x^r z^s dx}{\widehat{f}_{n,z}}, \quad 0 \leq r, s, r+s \leq n-5$$

and

$$\theta_i = (x^i - z^{n-i}) \frac{dx}{\widehat{f}_{n,z}}, \quad 3 \leq i \leq n-3$$

make up a basis of $H^0(C_n, \Omega^1) \simeq H^0(C_n^0, \Omega^1)$.

(2) *The elements*

$$\widetilde{\omega}_{r,s} := \frac{wx^r z^s dx}{\widehat{f}_{n,z}}, \quad 0 \leq r, s, r+s \leq n-4$$

and

$$\widetilde{\theta}_i = (x^i - z^{n-i}) \frac{dx}{w\widehat{f}_{n,z}}, \quad 2 \leq i \leq n-2$$

with a basis of $H^0(C_n, \Omega^1)$ as above make up a basis of $H^0(\widetilde{C}_n, \Omega^1) \simeq H^0(\widetilde{C}_n^0, \Omega^1)$. In particular

$$\langle \widetilde{\omega}_{r,s}, \widetilde{\theta}_i \mid 0 \leq r, s, r+s \leq n-4, 2 \leq i \leq n-2 \rangle = H^0(\widetilde{C}_n, \Omega^1)^{\gamma^* = -1} \simeq H^0(J_n, \Omega_{J_n}^1).$$

Here the superscript $\gamma^* = -1$ means the maximal subspace so that the pullback γ^* of γ acts as the multiplication by -1 .

Proof. Put $\omega = \frac{dx}{\widehat{f}_{n,z}}$. In what follows we compute the divisor of ω . We will take a careful analysis around the singularities on C_n^0 .

Let P be a non-singular point on the affine model C_n^0 . Then $\text{ord}_P(\omega) = 0$ since $\frac{dx}{\widehat{f}_{n,z}} = -\frac{dz}{\widehat{f}_{n,x}}$ and $(\widehat{f}_{n,z}(P), \widehat{f}_{n,x}(P)) \neq (0, 0)$. Let R be the strict transform of n -nodes P_0, \dots, P_{n-1} for the normalization $\pi : C_n \rightarrow C_n^0$. It consists of $2n$ points $\{Q_i, \beta(Q_i) \mid i = 0, \dots, n-1\}$ which will be specified later. We first consider the point $P_0 = (1 : 1 : 1)$. Put $x_1 = x + 1$ and $z_1 = z + 1$. Consider the curve defined by

$$f_n(x_1 + 1, 1, z_1 + 1) = 0$$

and blow up it along $P'_0 := (0, 0)$ in (x_1, z_1) -plane $\mathbb{A}_{(x_1, z_1)}^2$. We have an affine (isomorphic) model of a desingularization of C_n^0 at P_0 by gluing

$$U_{x_1} : \left\{ \begin{array}{l} z_1 = x_1 s \\ f(x_1, s) = 0 \end{array} \right. \subset \mathbb{A}_{(x_1, z_1, s)}^3 \text{ and } U_{z_1} : \left\{ \begin{array}{l} x_1 = z_1 t \\ g(z_1, t) = 0 \end{array} \right. \subset \mathbb{A}_{(x_1, z_1, t)}^3$$

with the relation $st = 1$. Here f is defined by the relation $f_n(1 + x_1, 1, 1 + x_1 s) = x_1^2 f(x_1, s)$. It is easy to see that $f(x_1, s)$ takes the following form

$$f(x_1, s) = p_0 + p_1 s + p_0 s^2 + x_1 q(x_1, s), \quad p_0, p_1 \in k[a_i] \setminus \{0\}, \quad q(x_1, s) \in k[x_1, s].$$

By symmetry we have $g(x_1, s) = f(x_1, s)$. Then we may put $Q_0 = (x_1, z_1, s) = (0, 0, \alpha_1)$ where α_1 is a root of $p_0 + p_1 s + p_0 s^2 = 0$ with another root α_2 and then

we have $\beta(Q_0) = (0, 0, \alpha_2)$. Since $\omega = \frac{dx_1}{x_1(p_1 + 2p_0 s + x_1 \frac{\partial q}{\partial s}(x_1, s))}$, if we choose

x_1 as a local parameter at Q_0 , then we see that $\text{ord}_{Q_0}(\omega) = -1$. Therefore for any $r, s \geq 0$, the differential form $\omega_{r,s} = (xz-1)x^r z^s \omega$ is holomorphic at Q_0 and $\beta(Q_0)$ because of the factor $xz-1$. By a similar argument one can treat other points in

R though the symmetry for β would be violated. So we omit details. What we remain to do is to check the order at infinity points.

Let $x = \frac{1}{u}$ with a local parameter u at an infinity point. Note that there are exactly n infinity points and they are corresponding to $(x : 0 : z)$ satisfying $f_n(x, 0, z) = 0$. It is easy to check that $\text{ord}_u(z) = -1$ since $f_n(x, 0, z) = x^n + z^n + (\text{lower terms}) = 0$. One can also check $\text{ord}_u(\omega) = n - 3$ by direct computation. Therefore we have

$$\text{ord}_u(\omega_{r,s}) = -2 - r - s + n - 3 = n - 5 - (r + s) \geq 0$$

by the assumption on r, s .

For θ_i , the factor $x^i - z^{n-i}$ vanishes at all P_j , $0 \leq j \leq n - 1$ (and also at all points in R). Therefore we may consider the order at infinity points. The details are omitted. Summing up we have

$$\frac{(n-4)(n-3)}{2} + n - 5 = \frac{n^2 - 5n + 2}{2} = g(C_n)$$

elements which give a basis of $H^0(C_n, \Omega^1)$.

For the second claim the computation would be similar. So we only give key points. Let u be a local parameter of a point P in R . Since the double cover $\tilde{C}_n \rightarrow C_n, w^2 = 1 - xz$ is ramified at P , one has $\text{ord}_u(x) = 2$ and $\text{ord}_u(w) = 1$. Therefore we have

$$\text{ord}_u(w \frac{dx}{f_z}) = 1 + 1 - 2 = 0, \text{ord}_u(\frac{dx}{w}) = 1 - 1 = 0.$$

These imply the holomorphy for elements $\tilde{\omega}_{r,s}, \tilde{\theta}_i$ in the claim with the same analysis at infinite points. We have

$$\frac{(n-3)(n-2)}{2} + n - 3 = \frac{n^2 - 3n}{2} = g(\tilde{C}_n) - g(C_n)$$

elements so that γ^* acts as the multiplication by -1 . Hence they give a basis of $H^0(\tilde{C}_n, \Omega^1)^{\gamma^* = -1}$. \square

4. DECOMPOSITION OF JACOBIANS J_n .

In this section we determine the subcovers of C_n^0 and \tilde{C}_n^0 fixed by their automorphisms and their smooth models. We start to work with C_n^0 to compute the defining equations of quotient curves but consider a smooth model at the same time. Therefore we often state our results for a smooth model of a singular curve appear here.

In what follows we consider the following quotient curves. for any $n \geq 4$ we put

$$\mathcal{X}_n := C_n / \langle \alpha \rangle, \tilde{\mathcal{X}}_n := \tilde{C}_n / \langle \alpha \rangle, \mathcal{Y}_n := C_n / \langle \beta \rangle, \tilde{\mathcal{Y}}_n := \tilde{C}_n / \langle \beta \rangle$$

For even n we write $n = 2^k m \geq 6$ with $k \geq 1$ and odd m . Put

$$\mathcal{Z}_n := C_n / \langle \beta \alpha^{\frac{n}{m}} \rangle, \tilde{\mathcal{Z}}_n := \tilde{C}_n / \langle \beta \alpha^{\frac{n}{m}} \rangle.$$

Finally for $n = 4$, put

$$\mathcal{W}_4 := C_4 / \langle \beta \alpha^2 \rangle, \tilde{\mathcal{W}}_4 := \tilde{C}_4 / \langle \beta \alpha^2 \rangle.$$

It will turn out that $\tilde{\mathcal{X}}_n$ is a hyperelliptic curve with the hyperelliptic involution δ . Then we put

$$\tilde{\mathcal{E}}_n := \tilde{\mathcal{X}}_n / \langle \delta \gamma \rangle = \tilde{C}_n / \langle \alpha, \delta \gamma \rangle.$$

Note that the automorphisms other than α are all involutions in each case. We also introduce the singular version $\mathcal{X}_n^0, \tilde{\mathcal{X}}_n^0, \dots$ etc in the same manner as above. Hence we have obtained the following figures:

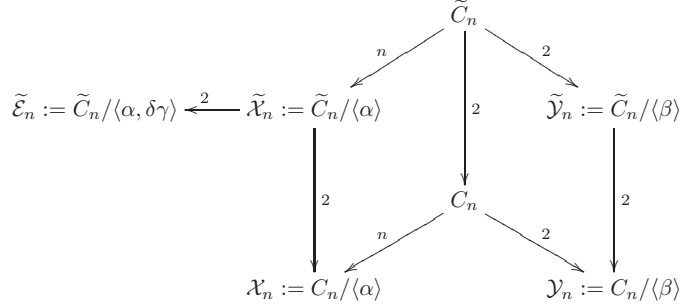


FIGURE 1. The diagram of maps

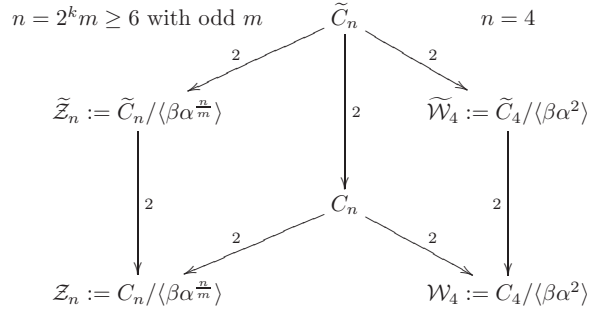


FIGURE 2. The diagram of maps

At first we start with the subcovers fixed by α .

Proposition 4.1. *i) The curve \mathcal{X}_n^0 has the following affine model*

$$\mathcal{X}_n^0 : \quad s^2 + s \left(n(1-t) - 2 + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (t^i - it + i - 1) \right) + t^n = 0.$$

ii) The smooth model \mathcal{X}_n of \mathcal{X}_n^0 is a hyperelliptic (or elliptic) curve of generic genus $g = \lfloor \frac{n-3}{2} \rfloor$ defined by

$$\mathcal{X}_n : \quad u^2 = f_n^0(t)$$

where $f_n^0(t) = \frac{1}{(t-1)^2} \left(\frac{A^2}{4} - t^n \right) \in k[t]$ and $A := n(1-t) - 2 + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (t^i - it + i - 1)$.

iii) The smooth model $\tilde{\mathcal{X}}_n$ of $\tilde{\mathcal{X}}_n^0$ is a hyperelliptic curve of generic genus $g = n - 3$ defined by

$$\tilde{\mathcal{X}}_n : \quad u^2 = f_n^0(1 - w^2)$$

iv) The smooth model $\tilde{\mathcal{E}}_n$ of $\tilde{\mathcal{E}}_n^0$ is a hyperelliptic curve of generic genus $g = \lfloor \frac{n-2}{2} \rfloor$ defined by

$$\tilde{\mathcal{E}}_n : U^2 = W f_n^0(1 - W)$$

Proof. We start with the curve C_n^0 and its automorphism α . It is obvious from Eq. (1) that α fixes $s = x^n$ and $t = xz$. Thus, the equation of $f_n(x, y, z) = 0$ gives

$$x^{2n} + t^n + x^n \left(y^n(n-2) - ny^{n-2}t + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (t^i y^{n-2i} - it y^{n-2} + (i-1)y^n) \right) = 0$$

Putting $y = 1$ we have an affine model

$$\mathcal{X}_n^0 : s^2 + s \left(n - 2 - nt + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (t^i - it + i - 1) \right) + t^n = 0$$

in the first claim. Denote by

$$A = \left(n(1-t) - 2 + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (t^i - it + i - 1) \right).$$

By making the substitution $u' = s + \frac{A}{2}$ we get $u'^2 = \frac{A^2}{4} - t^n$. Since $\frac{d^i}{dt^i} \left(\frac{A^2}{4} - t^n \right) \Big|_{t=1} = 0$ for $i = 0, 1$, the polynomial $\frac{A^2}{4} - t^n$ is divisible by $(t-1)^2$. Put

$$f_n^0(t) := \frac{1}{(t-1)^2} \left(\frac{A^2}{4} - t^n \right)$$

as in the statement. The map $C_n \rightarrow \mathcal{X}_n$ is unramified everywhere if n is odd while it is ramified exactly at n -points $(x : 0 : z)$, $f_n(x, 0, z) = 0$ with the ramification index 2 when n is even. Note that such n points are fixed by $\alpha^{\frac{n}{2}}$. It follows from this that the genus of \mathcal{X}_n (resp. $\tilde{\mathcal{X}}_n$) is $\lfloor \frac{n-3}{2} \rfloor$ (resp. $n-3$). This means that $f_n^0(t)$ is a separable polynomial of degree $n-2$ in t . Substitute u' with $u = \frac{u'}{(t-1)^2}$ a generic member \mathcal{X}_n^0 defined by

$$\mathcal{X}_n^0 : u^2 = f_n^0(t)$$

is a hyperelliptic curve of genus $\lfloor \frac{n-3}{2} \rfloor$.

Since the covering $\tilde{\mathcal{X}}_n \rightarrow \mathcal{X}_n$ is given by $w^2 = 1 - xz = 1 - t$, one has

$$\tilde{\mathcal{X}}_n : u^2 = f_n^0(1 - w^2)$$

with the hyperelliptic involution $\delta : (u, w) \mapsto (-u, w)$. Then clearly

$$\delta \gamma : (u, w) \mapsto (-u, -w)$$

and this gives rise to a model of $\tilde{\mathcal{E}}_n$ as below since $\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{E}}_n = \tilde{C}_n / \langle \alpha, \delta \gamma \rangle = \tilde{\mathcal{X}} / \langle \delta \gamma \rangle$:

$$\tilde{\mathcal{E}}_n : U^2 = W f_n^0(1 - W), (W, U) = (w^2, uw).$$

□

Summing up we have obtained the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde{C}_n & \xrightarrow[n:1]{\text{etale}} & \tilde{\mathcal{X}}_n & \xrightarrow{2:1} & \tilde{\mathcal{E}}_n, & n^2 - 4n + 1 & \longrightarrow & n - 3 & \longrightarrow & \lfloor \frac{n-2}{2} \rfloor \\
2:1 \downarrow & & 2:1 \downarrow & & & \downarrow & & \downarrow & & \\
C_n & \xrightarrow{n:1} & \mathcal{X}_n & & & \frac{n^2-5n+2}{2} & \longrightarrow & \lfloor \frac{n-3}{2} \rfloor.
\end{array}$$

Next we determine equations for the quotient curves \mathcal{Y}_n^0 and $\tilde{\mathcal{Y}}_n^0$ and their genii. Let $P_n(u, v)$ be the two variable polynomial over \mathbb{Z} so that $P_n(x+z, xz) = x^n + z^n$.

Proposition 4.2. *i) The curve \mathcal{Y}_n^0 has genus $g(\mathcal{Y}_n^0) = \begin{cases} \frac{(n-1)(n-5)}{4} & n \text{ is odd} \\ \frac{(n-2)(n-4)}{4} & n \text{ is even} \end{cases}$ and it has an affine model*

$$P_n(u, v) + (n-2) - nv + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i(v^i - iv + i - 1) = 0$$

ii) The curve $\tilde{\mathcal{Y}}_n^0$ has genus $g(\tilde{\mathcal{Y}}_n^0) = \begin{cases} \frac{(n-1)(n-4)}{2} & n \text{ is odd} \\ \frac{(n-2)(n-3)}{2} & n \text{ is even} \end{cases}$ and it has an affine model

$$P_n(u, 1-w^2) - 2 + nw^2 + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i\{(1-w^2)^i - 1 - iw^2\} = 0$$

Proof. Let us introduce the functions

$$u = x + z \quad \text{and} \quad v = xz$$

which are fixed by β . Then we can easily obtain the defining equations in the both claims.

We now compute the genus only for \mathcal{Y}_n^0 because the other case would be done similarly. In the case when n is odd, if $P = (x : y : z)$ is fixed by β , then it has to be

$$P = (1 : 0 : -1) \text{ or } P = (x : 1 : x).$$

In the latter case we see that $f_n(x, 1, x) = (x-1)^2 h(x)$ for some polynomial $h(x) \in k[x]$ with $h(\zeta^i) \neq 0$ for any $0 \leq i \leq n-1$ and $\deg_x h(x) = n-2$. Under normalization of C_n^0 , the proper transform of the point $(1 : 1 : 1)$ consists of two points interchanged by β . Therefore the number of fixed points of β is $1 + (n-2) = n-1$. By the Hurwitz formula for the double covering $C_n^0 \rightarrow \mathcal{Y}_n^0$ defined by u, v , we have the claim.

In the case when n is even, the fixed points consist of $P = (x : 1 : x)$, $x \neq \pm 1$. In fact we see that $f_n(x, 1, x) = (x^2 - 1)^2 h_1(x)$ for some polynomial $h_1(x) \in k[x]$ with $h(\zeta^i) \neq 0$ for any $0 \leq i \leq n-1$ and $\deg_x h_1(x) = n-4$. The claim follows by the Hurwitz formula again.

For $\tilde{\mathcal{Y}}_n^0$, in case when n is odd, the point $(1 : 0 : -1 : \pm 1)$ is no longer a fixed point of β just because of the new parameter w and then the number of fixed points are $2(n-2)$. In case when n is even, it is $2(n-4)$. The formula follows from these data. \square

Since the geometric genus is stable under the normalization, we have obtained the following commutative diagram:

$$\begin{array}{ccc}
\tilde{C}_n & \xrightarrow{2:1} & \tilde{\mathcal{Y}}_n = \tilde{C}_n / \langle \beta \rangle, & n^2 - 4n + 1 & \longrightarrow & \begin{cases} \frac{(n-1)(n-4)}{2} & (n \text{ is odd}) \\ \frac{(n-2)(n-3)}{2} & (n \text{ is even}) \end{cases} \\
\downarrow 2:1 & & \downarrow 2:1 & \downarrow & & \downarrow \\
C_n & \xrightarrow{2:1} & \mathcal{Y}_n = C_n / \langle \beta \rangle & \frac{n^2 - 5n + 2}{2} & \longrightarrow & \begin{cases} \frac{(n-1)(n-5)}{4} & (n \text{ is odd}) \\ \frac{(n-2)(n-4)}{4} & (n \text{ is even}) \end{cases}
\end{array}$$

Let $n = 2^k m \geq 6$ with $k \geq 1$ and m an odd number. Before giving the proof of the main theorem we observe an automorphism

$$\beta \alpha^{\frac{n}{m}} : (x : y : z) \mapsto (\zeta_{2^k}^{-1} z : y : \zeta_{2^k} x).$$

Clearly it is an involution. Recall the quotient curves $\tilde{\mathcal{Z}}_n = \tilde{C}_n / \langle \beta \alpha^{\frac{n}{m}} \rangle$ and $\mathcal{Z}_n = C_n / \langle \beta \alpha^{\frac{n}{m}} \rangle$. Since $2^k | n$, there exists a polynomial $Q_n(X, Y) \in k[X, Y]$ such that

$$Q_n(x + \zeta_{2^k}^{-1} z, xz) = x^n + z^n.$$

Then we have the following result.

Proposition 4.3. *Let n be as above.*

(1) *The curve \mathcal{Z}_n has genus $g(\mathcal{Z}_n) = \frac{n^2 - 6n + 4}{4}$ and it has an affine model*

$$Q_n(u, v) + (n-2) - nv + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i (v^i - iv + i - 1) = 0$$

$$\text{where } (u, v) = (x + \zeta_{2^k}^{-1} z, xz).$$

(2) *The curve $\tilde{\mathcal{Z}}_n$ has genus $g(\tilde{\mathcal{Z}}_n) = \frac{n^2 - 5n + 2}{2}$ and it has an affine model*

$$Q_n(u, 1 - w^2) - 2 + nw^2 + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} a_i \{(1 - w^2)^i - 1 - iw^2\} = 0.$$

Proof. The number of the fixed points of the involution $\beta \alpha^{\frac{n}{m}}$ on C_n (resp. \tilde{C}_n) is n (resp. $2n$). These are given by $(x : 1 : \zeta_{2^k} x)$ (resp. $(x : 1 : \zeta_{2^k} x : w)$) satisfying $f_n(x, 1, \zeta_{2^k} x) = 0$ (resp. $f_n(x, 1, \zeta_{2^k} x) = 0, w^2 = 1 + x^2$). Note that any fixed point never intersects with n -nodes or its proper transform. The genus for each curve is obtained from Hurwitz formula. The defining equations can be obtained easily. \square

In the case when $n = 4$, let us consider the involution defined by

$$\alpha^2 \beta = \beta \alpha^2 : (x : y : z) \mapsto (-z : y : -x).$$

By direct calculation we have the following.

Proposition 4.4. *The curve $\tilde{\mathcal{W}}_4$ is of genus one and it is defined by*

$$4u^2 + u^4 - 4u^2 w^2 + 2w^4 + a_2 w^4 = 0$$

where $u = x - z$ and $w^2 = 1 - xz$.

Let us recall the quotient map $\tilde{C}_n \rightarrow \tilde{\mathcal{Y}}_n$. It's pullback induces $\text{Jac}(\tilde{\mathcal{Y}}_n) \rightarrow \text{Jac}(\tilde{C}_n) \xrightarrow{1-\gamma^*} J_n$ which factors through $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$. Hence we have a natural map $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n) \rightarrow \text{Jac}(\tilde{C}_n)$. Similarly we also have a natural map $\text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n) \rightarrow \text{Jac}(\tilde{C}_n)$. The quotient maps $\tilde{C}_n \rightarrow \tilde{\mathcal{X}}_n \rightarrow \tilde{\mathcal{E}}_n$ induce a natural map $\text{Jac}(\tilde{\mathcal{E}}_n) \rightarrow J_n$ as well. The following fact is necessary to prove the main theorem.

Proposition 4.5. (1) *The natural map $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n) \rightarrow J_n$ induces an identification*

$$H^0(\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n), \Omega^1) \simeq \langle \tilde{\omega}_{r,s} - \tilde{\omega}_{s,r}, \tilde{\theta}_i + \beta^*(\tilde{\theta}_i) \mid 0 \leq r \leq s \leq \lfloor \frac{n-4}{2} \rfloor \rangle$$

(2) *The natural map $\text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n) \rightarrow J_n$ induces an identification*

$$H^0(\text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n), \Omega^1) \simeq \langle \tilde{\omega}_{r,s} + (\beta\alpha^{\frac{n}{m}})^*(\tilde{\omega}_{s,r}), \tilde{\theta}_i + (\beta\alpha^{\frac{n}{m}})^*(\tilde{\theta}_i) \mid 0 \leq r \leq s \leq \lfloor \frac{n-4}{2} \rfloor \rangle$$

(3) *The natural map $\text{Jac}(\tilde{\mathcal{E}}_n) \rightarrow J_n$ induces an identification*

$$H^0(\text{Jac}(\tilde{\mathcal{E}}_n), \Omega^1) \simeq \left\langle w(xz-1)^j \frac{dx}{\widehat{f}_{n,z}} \mid 0 \leq j \leq \lfloor \frac{n-4}{2} \rfloor \right\rangle.$$

Proof. The first claim follows directly from Theorem 3.1. For the second claim, we have to remark that when $i = \frac{n}{2}$,

$$\theta_{\frac{n}{2}} + (\beta\alpha^{\frac{n}{m}})^*(\theta_{\frac{n}{2}}) = \{(1 + \zeta_{2^k}^{\frac{n}{2}})x^{\frac{n}{2}} - (1 + \zeta_{2^k}^{-\frac{n}{2}})z^{\frac{n}{2}}\} \frac{dx}{w\widehat{f}_{n,z}} = 0$$

since $\zeta_{2^k}^{\frac{n}{2}} = (-1)^m = -1$. Note that $(\beta\alpha^{\frac{n}{m}})^*\left(\frac{dx}{\widehat{f}_{n,z}}\right) = \frac{dz}{\widehat{f}_{n,x}} = -\frac{dx}{\widehat{f}_{n,z}}$.

For the third claim, notice that the natural map $\tilde{C}_n \rightarrow \tilde{\mathcal{X}}_n$ induces a natural identification

$$H^0(\tilde{\mathcal{X}}_n, \Omega^1) = H^0(\tilde{C}_n, \Omega^1)^{\alpha^*=1}$$

and it is generated by $w^i \frac{dw}{u}$, $i = 0, \dots, n-4$. Let us remark that $w^i \frac{dw}{u} = w^{i+1}(xz-1) \frac{dx}{\widehat{f}_{n,z}}$ since

$$u = \frac{s + \frac{A}{2}}{(xz-1)^2} = \frac{x\widehat{f}_{n,x} - z\widehat{f}_{n,z}}{2(xz-1)^2}$$

and $2wdw = -zdx - xdz = (x\widehat{f}_{n,x} - z\widehat{f}_{n,z}) \frac{dx}{\widehat{f}_{n,z}}$. The claim follows from $H^0(\tilde{\mathcal{E}}_n, \Omega^1) = H^0(\tilde{\mathcal{X}}_n, \Omega^1)^{\gamma^*=-1} = H^0(\tilde{\mathcal{X}}_n, \Omega^1)^{(\delta\gamma)^*=1}$. \square

We now give a structure theorem for a decomposition of the Prym variety $J_n := \text{Prym}(\tilde{C}_n/C_n)$:

Theorem 4.6. *Assume $n \geq 4$.*

(1) *If n is odd, then*

$$J_n \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_n) \times \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)^2$$

where the factor $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$ is an abelian variety over k whose endomorphism ring contains $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Here $\zeta_n \in \mathbb{C}$ is a primitive n -th root of unity.

(2) If $n = 2^k m \geq 6$ with $k \geq 1$ and m an odd number, then

$$J_n \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_n) \times \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n) \times \text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n)$$

where the latter two Prym varieties are an abelian variety over k whose endomorphism ring contains $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ respectively.

(3) If $n = 4$, then

$$J_4 \stackrel{k}{\sim} \text{Prym}(\tilde{\mathcal{Y}}_4/\mathcal{Y}_4) \times \text{Prym}(\tilde{\mathcal{W}}_4/\mathcal{W}_4)$$

Proof. Let A be the connected component of $\text{Ker}(1 - \alpha^* : J_n \rightarrow J_n)$. The inclusion $A \subset J_n$ and the map $\text{Jac}(\tilde{\mathcal{E}}_n) \rightarrow J_n$ explained before induces an identification

$$H^0(A, \Omega^1) = H^0(J_n, \Omega^1)^{\alpha^*=1} = H^0(\tilde{\mathcal{X}}_n, \Omega^1)^{\gamma^*=-1} = H^0(\tilde{\mathcal{E}}_n, \Omega^1)$$

which implies that $A \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_n)$.

Put $B = J_n/A$. Since $\beta\alpha\beta = \alpha^{-1}$, the pullback β^* acts on B . By Theorem 3.1 one can see that

$$H^0(B, \Omega^1) = H^0(\tilde{C}_n, \Omega^1)^{\alpha^* \neq 1, \gamma^* = -1} = \langle \tilde{\omega}_{r,s}, \tilde{\theta}_i \mid \begin{matrix} 0 \leq r, s, r+s \leq n-4, r \neq s \\ 2 \leq i \leq n-2 \end{matrix} \rangle$$

where α^* acts by

$$\alpha^*(\tilde{\omega}_{r,s}) = \zeta^{r-s} \tilde{\omega}_{r,s}, \quad \alpha^*(\tilde{\theta}_i) = \zeta^i \tilde{\theta}_i.$$

By using this basis when n is odd one can check that the number of eigenvectors for β^* with the eigenvalue $+1$ is same as those with the eigenvalue -1 . Hence the connected component B_{\pm} of the kernel of the map $1 \pm \beta^* : B \rightarrow B$ is an abelian variety of dimension $\frac{1}{2}\dim B$ respectively and we have $B_+ \times B_- \stackrel{k}{\sim} B$. The inclusion $B_- \subset B$ and the composition $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n) \rightarrow J_n \rightarrow B$ induce an isogeny $B_- \stackrel{k}{\sim} \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$ since $H^0(B_-, \Omega^1) = H^0(B, \Omega^1)^{\beta^*=1} = H^0(\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n), \Omega^1)$ by Proposition 4.5-(1). By looking the action on $H^0(B_-, \Omega^1)$ we see that α never preserves B_- and this fact gives rise to an isogeny $B_- \stackrel{k}{\sim} B_+$. Since $\alpha^* + (\alpha^{-1})^*$ commutes with β^* , it acts on B_{\pm} . Hence,

$$\text{End}_k(\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{End}_k(B_{\pm}) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}(\zeta_n + \zeta_n^{-1}).$$

This proves the first claim.

For the second claim we consider the connected component B'_{\pm} of the kernel of the map $1 \pm (\beta\alpha^{\frac{n}{m}})^* : B \rightarrow B$. Then natural maps $B'_- \subset B \leftarrow J_n \leftarrow \text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n)$ imply an isogeny $B'_- \stackrel{k}{\sim} \text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n)$ since

$$H^0(B'_-, \Omega^1) = H^0(B, \Omega^1)^{(\beta\alpha^{\frac{n}{m}})^*=1} = H^0(\text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n), \Omega^1).$$

Since

$$1 - \beta^* = 1 - (\beta\alpha^{\frac{n}{m}}) = (1 - (\beta\alpha^{\frac{n}{m}})^*) \sum_{j=0}^{m-1} ((\beta\alpha^{\frac{n}{m}})^j)^*$$

,

$B'_+ \subset B \leftarrow J_n \xleftarrow{\alpha^*} \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$ imply an isogeny $B'_+ \stackrel{k}{\sim} \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$ since

$$H^0(B'_+, \Omega^1) = H^0(B, \Omega^1)^{(\beta\alpha^{\frac{n}{m}})^*=-1} = H^0(\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n), \Omega^1).$$

We can easily see that $H^0(B, \Omega^1)^{\beta^*=1}$ is a complement of $H^0(B'_+, \Omega^1)$ in $H^0(B, \Omega^1)$ and this implies that $\text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n)$ is also a complement of B_- in B under the isogeny induced by the previous argument for odd n . Hence we have

$$J_n \stackrel{k}{\sim} B_+ \times B_- \stackrel{k}{\sim} \text{Prym}(\tilde{\mathcal{Y}}_n/\mathcal{Y}_n) \times \text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n).$$

It follows from the same reason as above that

$$\text{End}_k(\text{Prym}(\tilde{\mathcal{Z}}_n/\mathcal{Z}_n)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{End}_k(B_+) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}(\zeta_n + \zeta_n^{-1}).$$

The last claim is just a direct computation and so omitted. This completes the proof. \square

5. EXPLICIT COMPUTATIONS FOR $n = 4, 5, 6, 7, 8$

In this section we explicitly decompose J_n for $4 \leq n \leq 8$. Most of the computations are straight forward and we skip some of the details. We start with the case of $n = 5$ and put the case of $n = 4$ at the last part.

5.1. C_5^0 as the first interesting case. In the case when $n = 5$ this curve appears in previous study on a decomposition of the intermediate Jacobians for some cubic threefolds [GY] but it was not studied because of the singularities. Put $a = a_2$. Recall the defining equation:

$$(5) \quad C_5^0 : x^5 + z^5 + 3y^5 - 5xy^3z + a(x^2yz^2 - 2xy^3z + y^5) = 0$$

By Theorem 4.6 we see that

$$J_5 \stackrel{k}{\sim} E \times B^2,$$

where E is an elliptic curve over k and B is an abelian surface over k such that $\text{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains $\mathbb{Q}(\sqrt{5})$. In what follows we will compute the equations of E and a hyperelliptic curve C whose Jacobian is isogenous over k to B . The strategy is similar to [GY]. For each curve appears here we work on a smooth model instead of the singular model. By using the automorphisms defined in (4) we have the commutative diagram

$$\begin{array}{ccccc} \tilde{C}_5 & \xrightarrow{5:1} & \tilde{C}_5/\alpha & \xrightarrow{2:1} & E := \tilde{C}_5/\langle \alpha, \delta\gamma \rangle, & 6 & \longrightarrow & 2 & \longrightarrow & 1 \\ 2:1 \downarrow & & 2:1 \downarrow & & & \downarrow & & \downarrow & & \\ C_5 & \xrightarrow{5:1} & C_5/\alpha & & & 1 & \longrightarrow & 1 & & \end{array}$$

where the right diagram indicates the genus of each curve on the left side. The vertical arrows on the left hand side stand for the quotient map by γ and the upper right arrow to E means the quotient map by the group $\alpha, \delta\gamma$ where δ is the hyperelliptic involution of \tilde{C}_5/α . As we will see below the curve \tilde{C}_5/α turns out to be hyperelliptic.

Explicit equations of the curves in the diagram are obtained step by step. For simplicity we let $y = 1$. Then

$$s := x^5 \quad \text{and} \quad t := xz$$

are fixed by α and this gives rise to

$$(6) \quad s^2 + (a(t-1)^2 - 5t + 3)s + t^5 = 0.$$

Substituting s by

$$u = (1 - t)s/2 + (a(t - 1)^2 - 5t + 3)/2,$$

we obtain

$$C_5/\alpha : \quad u^2 = f_5^0(t),$$

where

$$f_5^0(t) := -t^3 + \frac{1}{4}(a^2 - 8)t^2 - \frac{1}{2}(a^2 + 5a + 6)t + \frac{1}{4}(a + 3)^2.$$

Since the vertical arrow in the above diagram is given by $w^2 = 1 - xz = 1 - t$, one has

$$\tilde{C}_5/\alpha : \quad u^2 = f_5^0(1 - w^2)$$

with the hyperelliptic involution $\delta : (s, w) \mapsto (-s, w)$. Then clearly

$$\delta\gamma : (u, w) \mapsto (-u, -w)$$

and this gives rise to the smooth model E as below:

$$E : \quad y^2 = -\frac{1}{4}(4a + 15)x^3 + \frac{5}{2}(a + 4)x^2 + \frac{1}{4}(a^2 - 20)x + 1,$$

where $(x, y) = \left(\frac{1}{w^2}, \frac{u}{w^3}\right)$. By construction we see that

Proposition 5.1. *The elliptic curve E is a k -simple factor of J_5 .*

Similarly we consider the following commutative diagram but this time we use β instead of α :

$$\begin{array}{ccccc} \tilde{C}_5 & \xrightarrow{2:1} & \tilde{C}_5/\beta, & 6 & \longrightarrow & 2 \\ 2:1 \downarrow & & 2:1 \downarrow & \downarrow & & \downarrow \\ C_5 & \xrightarrow{2:1} & C_5/\beta & 1 & \longrightarrow & 0. \end{array}$$

The genus of \tilde{C}_5/β (resp. C_5/β) can be computed by counting fixed points of β on \tilde{C}_5 (resp. C_5). Clearly $T_1 = x + z$, $T_2 = xz$ are invariant under β and these are generators of the function field $k(C_5/\beta)$. Then we have an affine model of C_5/β :

$$3 + a + T_1^5 - 5T_2 - 2aT_2 - 5T_1^3T_2 + aT_2^2 + 5T_1T_2^2 = 0.$$

Since it is rational, we would find a rational parameter. Put

$$(7) \quad t_1 = \frac{-5 - 2a - 5T_1^3 + 2aT_2 + 10T_1T_2}{-1 + T_1 + T_1^2}.$$

Then one has

$$-25 - 8a + t_1^2 + 10T_1 + 4aT_1 - 5T_1^2 = 0$$

and it has a rational solution $(t_1, T_1) = (a + 5, a + 2)$. Putting $x = \frac{t_1 - a - 5}{T_1 - a - 2}$ and substituting this into the above equation we obtain

$$(8) \quad (t_1, T_1) = \left(\frac{25 + 5a - 10x - 6ax + 5x^2 + ax^2}{5 - x^2}, \frac{a - 10x - 2ax + 2x^2 + ax^2}{-5 + x^2} \right).$$

To gain the equation of \tilde{C}_5/β we substitute $w^2 = 1 - xz = 1 - T_2$ and (8) into (7). Simplifying the factors we obtain

$$w^2 = -\frac{(-10 - a + ax)g(x)}{(x^2 - 5)^2(-25 - 5a + 5x + 3ax)}, \quad g(x) = \sum_{i=0}^4 c_i x^i,$$

where

$$\begin{aligned} c_4 &= 5 + 5a + a^2, \\ c_3 &= -2(5 + a)(5 + 2a), \\ c_2 &= 2(50 + 20a + 3a^2), \\ c_1 &= -2(5 + a)(-5 + 2a), \\ c_0 &= -25 - 5a + a^2. \end{aligned}$$

We introduce a new parameter

$$y = (-25 - 5a + 5x + 3ax)(x^2 - 5)w$$

and then we have a birational model of \tilde{C}_5/β :

$$C : y^2 = h(x), \quad h(x) = -(3ax + 5x - 25 - 5a)(ax - a - 10)g(x).$$

The discriminant of $h(x)$ is

$$\Delta_h = 2^{30}5^5(15 + 4a)^2(-25 - 5a + a^2)^{14}.$$

Summing up by Theorem 4.6 we have.

Proposition 5.2. *Let $B = \text{Jac}(C)$. Then*

$$J_5 \stackrel{k}{\sim} E \times B^2$$

and $\text{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains $\mathbb{Q}(\sqrt{5})$.

5.2. The case $n = 6$. The genus 4 curve C_6^0 has equation as follows

$$(9) \quad \begin{aligned} x^6 + z^6 + 4y^6 - 6xyz^4 + a_2(x^2z^2y^2 - 2xyz^4 + y^6) \\ + a_3(x^3z^3 - 3xyz^4 + 2y^6) = 0 \end{aligned}$$

As in the previous case we work on a smooth model for each curve. By using the automorphisms defined in (4) we have the commutative diagram

$$\begin{array}{ccccccc} \tilde{C}_6 & \xrightarrow{6:1} & \tilde{\mathcal{X}}_6 & \xrightarrow{2:1} & \tilde{\mathcal{E}}_6 := \tilde{C}_6/\langle \alpha, \delta\gamma \rangle, & 13 & \longrightarrow & 3 & \longrightarrow & 2 \\ 2:1 \downarrow & & 2:1 \downarrow & & & \downarrow & & \downarrow & & \\ C_6 & \xrightarrow{6:1} & \mathcal{X}_6 & & & 4 & \longrightarrow & 1 & & \end{array}$$

By Proposition 4.1 we have an equation defining \mathcal{X}_6 :

$$\mathcal{X}_6 : \quad u^2 = f_6^0(t)$$

where $f_6^0(t)$ has degree 4 and is given as follows

$$(10) \quad \begin{aligned} f_6^0(t) &= \frac{1}{4}(-2 + a_3)(2 + a_3)t^4 + \frac{1}{2}(-4 + a_2a_3 + a_3^2)t^3 \\ &+ \frac{1}{4}(-12 + a_2^2 - 12a_3 - 3a_3^2)t^2 - \frac{1}{2}(2 + a_2 + a_3)(4 + a_2 + 2a_3)t \\ &+ \frac{1}{4}(4 + a_2 + 2a_3)^2 \end{aligned}$$

Since the vertical arrow in the above diagram is given by $w^2 = 1 - xz = 1 - t$, one has the genus 3 hyperelliptic curve

$$\tilde{\mathcal{X}}_6 : \quad u^2 = f_6^0(1 - w^2)$$

with the hyperelliptic involution $\delta : (u, w) \mapsto (-u, w)$. Then clearly

$$\delta \gamma : (u, w) \mapsto (-u, -w)$$

and this gives rise to an affine smooth model of $\tilde{\mathcal{E}}_6$ as below:

$$\begin{aligned} \tilde{\mathcal{E}}_6 : y^2 &= \frac{1}{4}(-2 + a_3)(2 + a_3)x^5 + \frac{1}{2}(12 - a_2a_3 - 3a_3^2)x^4 \\ (11) \quad &+ \frac{1}{4}(-60 + a_2^2 - 12a_3 + 6a_2a_3 + 9a_3^2)x^3 + (20 + 3a_2 + 10a_3)x^2 \\ &+ (-6 - a_2 - 3a_3)x \end{aligned}$$

where $(x, y) = (w^2, uw)$.

Next we determine the equations for the quotient curves \mathcal{Y}_6 and $\tilde{\mathcal{Y}}_6$ and their genii. Using the automorphisms defined in (4) we have the following diagram

$$\begin{array}{ccc} \tilde{C}_6 & \xrightarrow{2:1} & \tilde{\mathcal{Y}}_6 = \tilde{C}_6/\beta, & 13 & \longrightarrow & 6 \\ 2:1 \downarrow & & 2:1 \downarrow & \downarrow & & \downarrow \\ C_6 & \xrightarrow{2:1} & \mathcal{Y}_6 = C_6/\beta & 4 & \longrightarrow & 2. \end{array}$$

By Proposition 4.2 an defining equation of \mathcal{Y}_6 is given by

$$(12) \quad g_6(u, v) := u^6 - 6u^4v + 9u^2v^2 + a_3v^3 - 2v^3 + v^2a_2 - 2va_2 - 3va_3 - 6v + a_2 + 2a_3 + 4 = 0$$

Recall that this is of genus 2. We now try to get a Weierstrass model of Y_6 as follows. Observe that it has a singularity at $(u, v) = (1, 1)$ and introduce $(u_1, v_1) = (u - 1, v - 1)$. By blowing at $(u_1, v_1) = (0, 0)$ which corresponds to considering $v_1 = tu_1$ with a new parameter t . Then we have an equation $f(u_1, t) = 0$ which is degree 4 (resp. 3) in u_1 (resp. t). This equation has a singularity at $(u_1, t) = (-2, 0)$. We substitute $u_2 = u_1 + 2$ and take the blowing up $u_2 = u_3t$. Then we have an equation

$$Q_2(u_3)t^2 + Q_1(u_3)t = Q_0(u_3), \quad Q_i \in k[u_3], \quad \deg_{u_3} Q_i = 2 + i \text{ for } i = 0, 1, 2$$

which gives rise to a defining equation of \mathcal{Y}_6 . Substituting t with $\frac{t}{Q_2(u_3)}$ we will

get a hyperelliptic model. To be more precise, put $x = \frac{u^2 - 1}{v - 1}$ and

$$y = \frac{u(-2 - a_3 + 3u^4 + u^6 + (6 + 3a_3 - 6u^2 - 6u^4)v + (-3 - 3a_3 + 9u^2)v^2 - (2 + a_3)v^3)}{(v - 1)^3}.$$

Then we have an affine smooth model:

$$\mathcal{Y}_6 : y^2 = (x^3 - 6x^2 + 9x + a_3 - 2)(4x^3 - 12x^2 + 6x - (a_2 + 3a_3)x + a_3 - 2).$$

Next we consider a singular model of $\tilde{\mathcal{Y}}_6$

$$g_6(u, 1 - w^2) = 0.$$

This curve has two automorphisms

$$\tau_1 : (u, w) \mapsto (-u, w), \quad \tau_2 : (u, w) \mapsto (-u, -w).$$

Let us compute the equations of two curves $\tilde{\mathcal{Y}}_6/\langle \tau_i \rangle$, $i = 1, 2$ which turn out to be factors of $\text{Prym}(\tilde{\mathcal{Y}}_6/\mathcal{Y}_6)$. Consider the equation $g_6^1(u', w) = 0$ so that $g_6^1(u^2, w) = g_6(u, w)$ (hence we putted $u' = u^2$). Then, the curve defined by $g_6^1(u', w) = 0$ has

a singularity at $(u', w) = (1, 0)$ in the (u', w) -plane. Substitute $u_1 = u' - 1$ and set $u_1 = u_2 w^2$. Then we have the equation

$$Q_3^1(u_2)w^2 = Q_2^1(u_2), \quad Q_i \in k[u_2], \quad \deg_{u_2} Q_i^1 = i \text{ for } i = 2, 3.$$

Substituting t with $\frac{t}{Q_3^1(u_2)}$ we will have a hyperelliptic model. To be more precise, put Put

$$x = \frac{1 - u'}{w^2}, \quad y = \frac{(2 - a_3)w^6 + 9(u' - 1)w^4 + 6(u' - 1)^2 w^2 + (u' - 1)^3}{w^5}.$$

This gives rise to an affine smooth model of $\tilde{\mathcal{Y}}_6/\langle \tau_1 \rangle$:

$$\tilde{\mathcal{Y}}_6/\langle \tau_1 \rangle : y^2 = (3x^2 - 6x - 3 - a_2 - 3a_3)(x^3 - 6x^2 + 9x - 2 + a_3).$$

For the curve $\tilde{\mathcal{Y}}_6/\langle \tau_2 \rangle$ we consider two functions $s = uw$ and $t = u^2$ which is invariant under τ_2 . Consider the polynomial $g_6^2(s, t)$ so that $g_6^2(uw, u^2) = g_6(u, w)$. This curve has singularities at $(s, t) = (0, 0), (0, 1)$. Therefore put $s^2 = Xt(t - 1)$. Then, after deleting the factor $(t - 1)^2$ the polynomial $g_6^2(s, t) = 0$ implies

$$4 - t + (12 - 6t)X + (6 - 9t - a_2 - 3a_3)X^2 + (2 - a_3 - 2t + a_3t)X^3 = 0.$$

Solving this in t , we have

$$t = \frac{-4 - 12X + (-6 + a_2 + 3a_3)X^2 + (-2 + a_3)X^3}{-1 - 6X - 9X^2 + (-2 + a_3)X^3}.$$

Therefore we have

$$s^2 = Xt(t - 1) = \frac{X(-3 - 6X + (3 + a_2 + 3a_3)X^2)}{-1 - 6X - 9X^2 + (-2 + a_3)X^3}.$$

Put $Y = s(-1 - 6X - 9X^2 + (-2 + a_3)X^3)$ and then we have a hyperelliptic model

$$\tilde{\mathcal{Y}}_6/\langle \tau_2 \rangle : Y^2 = X\{(3 + a_2 + 3a_3)X^2 - 6X - 3\}\{(-2 + a_3)X^3 - 9X^2 - 1 - 6X - 1\}.$$

Next, is left to analyze $\tilde{\mathcal{Z}}_6$ and \mathcal{Z}_6 . For these curves we have the following diagram

$$\begin{array}{ccccc} \tilde{C}_6 & \xrightarrow{2:1} & \tilde{\mathcal{Z}}_6 = \tilde{C}_6/\langle \alpha^3 \beta \rangle, & 13 & \longrightarrow & 4 \\ 2:1 \downarrow & & 2:1 \downarrow & \downarrow & & \downarrow \\ C_6 & \xrightarrow{2:1} & \mathcal{Z}_6 = C_6/\langle \alpha^3 \beta \rangle & 4 & \longrightarrow & 1. \end{array}$$

Notice that the curve $\tilde{\mathcal{Z}}$ inherits two involutions

$$\sigma_1 : (u, w) \mapsto (-u, w), \quad \sigma_2 : (u, w) \mapsto (-u, -w).$$

By Proposition 4.3 an defining equation of \mathcal{Z}_6 is given by

$$(13) \quad \begin{aligned} h_6(u, v) &:= u^6 + 6u^4v + 9u^2v^2 + 2v^3 - 6v + 4 \\ &\quad + v^2a_2 - 2va_2 - 3va_3 - 6v + a_2 + 2a_3 + 4 = 0. \end{aligned}$$

Since it is of genus 1 we try to find an Weierstrass model for \mathcal{Z}_6 as follows. Consider the equation $h_6^1(u_1, v)$ so that $h_6^1(u^2, v) = h_6(u, v)$. Then the curve defined by $h_6^1(u_1, v) = 0$ is rational and it has the singularities at $(u_1, v) = (0, 1), (-3, 1)$ on

(u_1, v) -plane. Put $v = v_1 + 1$ and $v_1 = v_2 u_1 (u_1 + 3)$. Further we put $u_1 = \frac{t}{v_2}$ to remove the remaining singularities. Then we have

$$v_2 = \frac{-1 - 6t - 9t^2 - 2t^3 - a_3 t^3}{t(6 + 6t + a_2 + 3a_3 + 3a_3 t)}$$

and this gives

$$u^2 = u_1 = \frac{t}{v_2} = \frac{t^2(6 + 6t + a_2 + 3a_3 + 3a_3 t)}{-1 - 6t - 9t^2 - 2t^3 - a_3 t^3}.$$

Put $s = \frac{-1 - 6t - 9t^2 - 2t^3 - a_3 t^3}{t} u$. Then we have a smooth affine Weierstrass model

$$\mathcal{Z}_6 : s^2 = -\{(3a_3 + 6)t + a_2 + 3a_3 + 6\}\{(a_3 + 2)t^3 + 9t^2 + 6t + 1\}.$$

On the other hand

$$w^2 = 1 - v = -t\left(2 + \frac{t}{v_2}\right) = \frac{t(-3 - 18t + (-21 + a_2 + 3a_3)t^2)}{1 + 6t + 9t^2 + (2 + a_3)t^3}.$$

Put $w' = (1 + 6t + 9t^2 + (2 + a_3)t^3)w$ and then we have an affine smooth model of the quotient curve $\tilde{\mathcal{Z}}_6/\langle\sigma_1\rangle$:

$$\tilde{\mathcal{Z}}_6/\langle\sigma_1\rangle : w'^2 = t(-3 - 18t + (-21 + a_2 + 3a_3)t^2)(1 + 6t + 9t^2 + (2 + a_3)t^3).$$

The computation for $\tilde{\mathcal{Z}}_6/\langle\sigma_2\rangle$ will be similar to what we carried out for $\tilde{\mathcal{Y}}_6/\langle\tau_2\rangle$. Put $X - uw$ and $Y - w^2$. Then we have a singular model

$$(14) \quad h_6^2(X, Y) = X^6 + 6X^4Y + 9X^2Y^2 - 6X^4Y^2 - 18X^2Y^3 + 9X^2Y^4 + 6Y^5 - 2Y^6 + a_2Y^5 + 3a_3Y^5 - a_3Y^6 = 0.$$

Put $x_1 = X^2$ and $x_1 = Y^2x_2$ to remove the singularities. Further we put

$$Y = \frac{y_1}{-2 + 9x_2 - 6x_2^2 + x_2^3 - a_3}, \quad y_1 = y_2 - \frac{1}{2}(6 - 18x_2 + 6x_2^2 + a_2 + 3a_3).$$

The equation (14) gives us that a quadratic equation which has a solution $(x_2, y_2) = (0, \frac{1}{2}(6 + a_2 + 3a_3))$. Put $x_2 = t(y_2 - \frac{1}{2}(6 + a_2 + 3a_3))$. Then we have

$$X^2 = x_2 = \frac{tq(t)^3(-1 + 18t^2 + 3a_2t^2 + 9a_3t^2)}{p(t)^2}$$

where $q(t) = 6 + 36t + a_2 + 9a_2t + 3a_3 + 18a_3t$ and $p(t)$ is a polynomial of (generic) degree 3 in t . We substitute $y = X \frac{p(t)}{q(t)}$ and then obtain

$$\tilde{\mathcal{Z}}_6/\langle\sigma_2\rangle : y^2 = t\{(36 + 9a_2 + 18a_3)t + 3a_3 + a_2 + 6\}\{(18 + 3a_2 + 9a_3)t^2 - 1\}.$$

Summing up we have the following result.

Proposition 5.3. *Keep the notation as above. Then*

$$J_6 \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_6) \times \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_1\rangle) \times \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_2\rangle) \times \text{Jac}(\tilde{\mathcal{Z}}_6/\langle\sigma_1\rangle) \times \text{Jac}(\tilde{\mathcal{Z}}_6/\langle\sigma_2\rangle).$$

Proof. By Theorem 4.6 we have

$$J_6 \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_6) \times \text{Prym}(\tilde{\mathcal{Y}}_6/\mathcal{Y}_6) \times \text{Prym}(\tilde{\mathcal{Z}}_6/\mathcal{Z}_6).$$

Applying Corollary 2.4 to $H = \langle\tau_1, \tau_2\rangle$ for $\tilde{\mathcal{Y}}_6$ we have

$$\text{Jac}(\tilde{\mathcal{Y}}_6) \stackrel{k}{\sim} \text{Jac}(\mathcal{Y}_6) \times \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_1\rangle) \times \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_2\rangle)$$

since the genus of $\tilde{\mathcal{Y}}_6/H$ is zero. This implies $\text{Prym}(\tilde{\mathcal{Y}}_6/\mathcal{Y}_6) \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_1\rangle) \times \text{Jac}(\tilde{\mathcal{Y}}_6/\langle\tau_2\rangle)$. We have $\text{Prym}(\tilde{\mathcal{Z}}_6/\mathcal{Z}_6) \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{Z}}_6/\langle\sigma_1\rangle) \times \text{Jac}(\tilde{\mathcal{Z}}_6/\langle\sigma_2\rangle)$ as well. \square

5.3. The case $n = 7$. The genus 8 curve C_7 has equation as follows

$$(15) \quad x^7 + z^7 + 5y^7 - 7xyz^5 + a_2(x^2z^2y^3 - 2xzy^5 + y^7) + a_3(x^3z^3y - 3xzy^5 + 2y^7) = 0$$

By Proposition 4.1 we have the following diagram

$$\begin{array}{ccccc} \tilde{C}_7 & \xrightarrow{7:1} & \tilde{\mathcal{X}}_7 = \tilde{C}_7/\alpha & \xrightarrow{2:1} & \tilde{\mathcal{E}}_7 = \tilde{C}_7/\langle\alpha, \delta\gamma\rangle, & 22 & \longrightarrow & 4 & \longrightarrow & 2 \\ 2:1 \downarrow & & 2:1 \downarrow & & \downarrow & & \downarrow & & & \\ C_7 & \xrightarrow{7:1} & \mathcal{X}_7 = C_7/\alpha & & 8 & \longrightarrow & 2 \end{array}$$

By Proposition 4.1 we have

$$(16) \quad \begin{aligned} f_7^0(t) &= -t^5 + \frac{1}{4}(-8 + a_3^2)t^4 + \frac{1}{2}(-6 + a_2a_3 + a_3^2)t^3 \\ &+ \frac{1}{4}(-16 + a_2^2 - 14a_3 - 3a_3^2)t^2 - \frac{1}{2}(2 + a_2 + a_3)(5 + a_2 + 2a_3)t \\ &+ \frac{1}{4}(5 + a_2 + 2a_3)^2 \end{aligned}$$

Since the vertical arrow in the above diagram is given by $w^2 = 1 - xz = 1 - t$, one has

$$\tilde{\mathcal{X}}_7 : u^2 = f_7^0(1 - w^2)$$

with the hyperelliptic involution $\delta : (u, w) \mapsto (-u, w)$. Then clearly

$$\delta\gamma : (u, w) \mapsto (-u, -w)$$

and this gives rise to a smooth model of $\tilde{\mathcal{E}}_7$ as below:

$$(17) \quad \begin{aligned} \tilde{\mathcal{E}}_7 : y^2 &= \left(-7 + \frac{a_3^2}{4}\right)x^5 + \left(21 - \frac{1}{2}a_2a_3 + -\frac{3a_3^2}{2}\right)x^4 \\ &+ \left(-35 + \frac{a_2^2}{4} - \frac{7a_3}{2} + \frac{3}{2}a_2a_3 + \frac{9a_3^2}{4}\right)x^3 \\ &+ \left(35 + \frac{7a_2}{2} + \frac{23a_3}{2}\right)x^2 + \left(-\frac{35}{4} - a_2 - 3a_3\right)x \end{aligned}$$

where $(x, y) = (w^2, uw)$ and it is of genus $g = 2$.

Next we determine equations for the quotient curves \mathcal{Y}_7 and $\tilde{\mathcal{Y}}_7$ and their genii. Start with a singular model of C_7 which has the equation given as in (15). By using the automorphisms we have the following diagram

$$\begin{array}{ccccc} \tilde{C}_7 & \xrightarrow{2:1} & \tilde{C}_7/\beta, & 22 & \longrightarrow & 9 \\ 2:1 \downarrow & & 2:1 \downarrow & & \downarrow & \downarrow \\ C_7 & \xrightarrow{2:1} & C_7/\beta & 8 & \longrightarrow & 3. \end{array}$$

By Proposition 4.2 the curve $\mathcal{Y}_7 = C_7/\langle\beta\rangle$ is of genus 3 and it is defined by

$$(18) \quad \begin{aligned} g_7(u, v) &:= u^7 - 7u^5v + 14u^3v^2 - 7uv^3 + v^3a_3 + v^2a_2 \\ &- 2va_2 - 3va_3 - 7v + a_2 + 2a_3 + 5 = 0. \end{aligned}$$

The curve $\tilde{\mathcal{Y}}_7$ is given by $g_7(u, 1 - w^2) = 0$. Summing up we have proved the following.

Proposition 5.4. *Keep the notation as above. Then,*

$$J_7 \stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_7) \times J_{7,\beta}^2$$

where $\text{End}(J_{7,\beta}) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

It would be interesting to find a Jacobian which is isogenous to $J_{7,\beta}$. We expect there exists a smooth curve D of genus 3 (maybe non-hyperelliptic) so that $\text{Jac}(D)^2 \stackrel{\bar{k}}{\sim} J_{7,\beta}$ with $\text{End}_{\bar{k}}(\text{Jac}(D)) \otimes_{\mathbb{Z}} \mathbb{Q} \supset \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

5.4. **The case $n = 8$.** The genus 13 curve C_8 has the following equation:

$$(19) \quad x^8 + z^8 + 6 - 8xz + a_2(z^2x^2 - 2xz + 1) + a_3(z^3x^3 - 3xz + 2) + a_4(z^4x^4 - 4xz + 3) = 0.$$

By Proposition 4.1 we have the following diagram

$$\begin{array}{ccccc} \tilde{C}_8 & \xrightarrow{8:1} & \tilde{\mathcal{X}}_8 = \tilde{C}_8/\alpha & \xrightarrow{2:1} & \tilde{\mathcal{E}}_8 = \tilde{C}_8/\langle \alpha, \delta\gamma \rangle, & 33 & \longrightarrow & 5 & \longrightarrow & 3 \\ 2:1 \downarrow & & 2:1 \downarrow & & & \downarrow & & \downarrow & & \\ C_8 & \xrightarrow{8:1} & \mathcal{X}_8 = C_8/\alpha & & & 13 & \longrightarrow & 2 & & \end{array}$$

By Proposition 4.1 we have an affine smooth model

$$\mathcal{X}_8 : u^2 = f_8^0(t)$$

where $f_8^0(t)$ has degree 6 and is given as follows

$$f_8^0(t) = \frac{1}{4} (a_4t^2 + 2t^2 + a_3t + 2a_4t + 4t + a_2 + 2a_3 + 3a_4 + 6) \\ (a_4t^4 - 2t^4 + a_3t^3 + a_2t^2 - 2a_2t - 3a_3t - 4a_4t - 8t + a_2 + 2a_3 + 3a_4 + 6)$$

By Proposition 4.1 one has

$$\tilde{\mathcal{X}}_8 : u^2 = f_8^0(1 - w^2)$$

with the hyperelliptic involution $\delta : (u, w) \mapsto (-u, w)$ and it is of genus 5. The involution

$$\delta\gamma : (u, w) \mapsto (-u, -w)$$

gives rise to a smooth model $\tilde{\mathcal{E}}_8$ as below:

$$\tilde{\mathcal{E}}_8 : y^2 = xf_8^0(1 - x)$$

where $(x, y) = (w^2, wu)$ and it is of genus 3.

Next we determine the equations for the quotient curves $\mathcal{Y}_8, \tilde{\mathcal{Y}}_8, \mathcal{W}_8$, and $\tilde{\mathcal{W}}_8$. By Proposition 4.2 and 4.4 we have the following diagram:

$$\begin{array}{ccccccc} \tilde{\mathcal{Z}}_8 & \xleftarrow{1:2} & \tilde{C}_8 & \xrightarrow{2:1} & \tilde{\mathcal{Y}}_8 = \tilde{C}_8/\beta, & 13 & \longleftarrow & 33 & \longrightarrow & 15 \\ 2:1 \downarrow & & 2:1 \downarrow & & 2:1 \downarrow & \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_8 & \xleftarrow{1:2} & C_8 & \xrightarrow{2:1} & \mathcal{Y}_8 = C_8/\beta & 5 & \longleftarrow & 13 & \longrightarrow & 6. \end{array}$$

By Proposition 4.2 the curve $\mathcal{Y}_8 = C_8/\langle\beta\rangle$ is of genus 6 and it is defined by

$$(20) \quad \begin{aligned} g_8(u, v) &:= u^8 - 8v - 8u^6v + 20u^4v^2 - 16u^2v^3 + 2v^4 \\ &\quad + 6 + a_2 - 2a_2v + a_2v^2 + 2a_3 - 3a_3v \\ &\quad + a_3v^3 + 3a_4 - 4a_4v + a_4v^4 = 0 \end{aligned}$$

where $(u, v) = (x + z, xz)$. The curve $\tilde{\mathcal{Y}}_8$ is given by $g_8(u, 1 - w^2) = 0$. Recall that as in the case of $n = 6$, the curve $\tilde{\mathcal{Y}}_8$ admits two involutions

$$\tau_1 : (u, w) \mapsto (-u, w), \quad \tau_2 : (u, w) \mapsto (-u, -w).$$

Put $H = \langle\tau_1, \tau_2\rangle$. Then we have the quotient curve $\tilde{\mathcal{Z}}_8/H = \mathcal{Z}_8/\tau_3$ where $\tau_3 : (u, v) \mapsto (-u, v)$. It is easy to see that this curve has genus 2. By Corollary 2.4 we have

$$\text{Jac}(\tilde{\mathcal{Y}}_8) \times \text{Jac}(\tilde{\mathcal{Y}}_8/H)^2 \stackrel{k}{\sim} \text{Jac}(\mathcal{Y}_8) \times \text{Jac}(\tilde{\mathcal{Y}}_8/\langle\tau_1\rangle) \times \text{Jac}(\tilde{\mathcal{Y}}_8/\langle\tau_2\rangle)$$

which induces

$$\text{Prym}(\tilde{\mathcal{Y}}_8/\mathcal{Y}_8) \stackrel{k}{\sim} \text{Prym}((\tilde{\mathcal{Y}}_8/\langle\tau_1\rangle)/(\tilde{\mathcal{Y}}_8/H)) \times \text{Prym}((\tilde{\mathcal{Y}}_8/\langle\tau_2\rangle)/(\tilde{\mathcal{Y}}_8/H)).$$

On the other hand by Proposition 4.3 the curve $\mathcal{Z}_8 = C_8/\langle\beta\alpha\rangle$ is of genus 5 and it is defined by

$$\begin{aligned} &-6 - u^8 + 8v + 8\zeta_8^{-1}u^6v + 20\zeta_8^2u^4v^2 - 16\zeta_8u^2v^3 + 2v^4 \\ &-a_2(v-1)^2 - a_3(v-1)^2(v+2) - a_3(v-1)^2(v^2+2v+3) = 0 \end{aligned}$$

where $(u, v) = (x + \zeta_8^{-1}z, xz)$. Replacing u with ζ_{16}^3u we have a nicer form

$$(21) \quad \begin{aligned} h_8(u, v) &:= -6 + u^8 + 8v + 8u^6v + 20u^4v^2 + 16u^2v^3 + 2v^4 - a_2(v-1)^2 \\ &\quad - a_3(v-1)^2(v+2) - a_3(v-1)^2(v^2+2v+3) = 0. \end{aligned}$$

The curve $\tilde{\mathcal{Z}}_8$ with genus 13 is given by $h_8(u, 1 - w^2) = 0$. Notice that the curve $\tilde{\mathcal{Z}}_8$ admits two involutions

$$\kappa_1 : (u, w) \mapsto (-u, w), \quad \kappa_2 : (u, w) \mapsto (-u, -w).$$

Put $H' = \langle\kappa_1, \kappa_2\rangle$. Then we have an elliptic curve $E_8 := \tilde{\mathcal{Z}}_8/H'$ over k . As seen before we have

$$\text{Prym}(\tilde{\mathcal{Z}}_8/\mathcal{Z}_8) \stackrel{k}{\sim} \text{Prym}((\tilde{\mathcal{Z}}_8/\langle\kappa_1\rangle)/E_8) \times \text{Prym}((\tilde{\mathcal{Z}}_8/\langle\kappa_2\rangle)/E_8).$$

Summing up we have the following .

Proposition 5.5. *Keep the notation as above. Then,*

$$\begin{aligned} J_8 &\stackrel{k}{\sim} \text{Jac}(\tilde{\mathcal{E}}_8) \times \text{Prym}((\tilde{\mathcal{Y}}_8/\langle\tau_1\rangle)/(\tilde{\mathcal{Y}}_8/H)) \times \text{Prym}((\tilde{\mathcal{Y}}_8/\langle\tau_2\rangle)/(\tilde{\mathcal{Y}}_8/H)) \\ &\quad \times \text{Prym}((\tilde{\mathcal{Z}}_8/\langle\kappa_1\rangle)/E_8) \times \text{Prym}((\tilde{\mathcal{Z}}_8/\langle\kappa_2\rangle)/E_8) \end{aligned}$$

where the endomorphism rings tensored by \mathbb{Q} for the last four factors contain $\mathbb{Q}(\zeta_8 + \zeta_8^{-1}) = \mathbb{Q}(\sqrt{2})$.

Proof. The claim follows from Theorem 4.6 and the computation done above. \square

It would be interesting to find all Jacobian surfaces over k with real multiplication by $\sqrt{2}$ which are factors of the above Prym varieties.

5.5. C_4^0 . Finally we treat the case of $n = 4$ which is the most simplest case. So we give explicit equations without details. Recall the equation defining:

$$(22) \quad C_4^0 : x^4 + z^4 + 2y^4 - 4xy^2z + a_2(x^2z^2 - 2xy^2z + y^4) = 0$$

By Theorem 4.6 we see that

$$J_4 \stackrel{k}{\sim} E_1 \times E_2 \stackrel{k}{\simeq} E_1^2$$

where

$$E_1 : y^2 = (a_2 + 2)x^4 + 4x^2 + 1, \quad E_2 : y^2 = (a_2 + 2)x^4 - 4x^2 + 1$$

and the isomorphism $E_1 \simeq E_2$ is given by $x \mapsto \sqrt{-1}x$. Note that $\sqrt{-1} = \zeta_4 \in k$.

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